Analysis of a Nutation Damper for a Two-Degree-of-Freedom Gyroscope

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The dynamics of a two-degree-of-freedom gyroscope with a ball-in-tube nutation damper mounted in the gimbal is analyzed. The nonlinear coupled equations of motion are solved by using the method of multiple scales. The results show that, if the natural frequency of the ball-in-tube damper is tuned to equal the nutation frequency of the gyroscope with the damper removed, the wobble motion of the gyroscope will be eliminated most rapidly and the decay time constant of the nutation angle is a concave function of the damping coefficient. If the nutation damper is oriented perpendicular to the spin axis of the rotor, the residual precession due to the gravity effect can be avoided.

Introduction

THE gyroscope is one of the two basic components of all inertial navigation systems (the other is an accelerometer). It serves as directional reference in an inertial frame. Gyroscopes are also used as motion sensors for stabilizing the motion of ships, aircraft, and other mechanical systems. Its design and control has been an important technological problem for half a century. The two-degree-of-freedom gyroscope has a support that permits the spin axis to have two degrees of rotational freedom with respect to the case of the gyroscope. Such a device is shown schematically in Fig. 1. The Cartesian coordinate system X,Y,Z is fixed on the case of the gyroscope with origin O coinciding with the center of mass of the rotor. The system x,y,z is fixed on the inner gimbal. The inner gimbal can rotate freely about the x axis. The outer gimbal can rotate freely about the Z axis. In the navigation system, the spin axis of the rotor is used to indicate one of the directions of the axes of the navigational frame, i.e., north, east, and down. These physical directions rotate with respect to inertial space due to Earth rotation and vehicle motion. Therefore, the spin axis of the rotor must be rotatable on command. This can be done by applying a calibrated torque to the inner gimbal and the outer gimbal. The rotor will inevitably result in a coning motion (precession and nutation) of the spin axis about the angular momentum vector of the gyroscope (which is fixed in space in the absence of subsequent external torque) after a torque with finite duration has been applied. Therefore, the gyroscope will be of practical value only if the nutation and precession are damped.[1] The phenomenon of coning motion also appears for the gyroscope, which serves as a motion sensor. The elimination of the coning motion will not only enhance the gyroscope's accuracy, but will also reduce the loading of the control unit.[2]

The coning motion can be suppressed by attaching a vibration damper to the gyroscope at some appropriate place. Physically, the vibration damper might be any device that can dissipate the energy of coning motion of the rotor and convert the transverse angular momentum of the gyroscope into spin angular momentum. The spin-stabilized satellite will also result in such coning motion after a reorientation maneuver. The commonly used damping devices for removing this coning motion of the satellite are viscous ring nutation dampers,[3-5] ball-in-tube nutation dampers,[6,7] elastomer dampers,[8] etc. The viscous ring nutation damper has a significant effect only if it is partially filled and is mounted on the spinning body. The undesired feature of such a damper is that there exists a residual nutation angle if the initial conical angle (or nutation angle) of coning motion is large.[4] Therefore, it is inapplicable to the two-degree-of-freedom gyroscope. Schneider and Likins[6] analyzed the performance of the nutation damper and precession damper of the ball-in-tube type. Cochran and Thompson[7] analyzed the stability of a similar problem and provided an approximate analytical expression for the ratio of the energy dissipation rates of these two dampers. Both Refs. 6 and 7 solved the problem by using the energy-sink approximation.

The ball-in-tube damper, which is most often subjected to analysis because of the simplicity in modeling and solution, is considered here. It is installed in the gyro-element inner gimbal as shown in Fig. 1, since a stability problem will incur if the damper is mounted on the spinning rotor. The ball mass is assumed to be sufficiently small in comparison with that of the rotor, and so the ratio of the moment of inertia of the ball with respect to the spin axis to the spin moment of inertia of the rotor is defined as a small parameter ϵ. The exact equations of motion, which are parameterized in ϵ, rather than the ap-

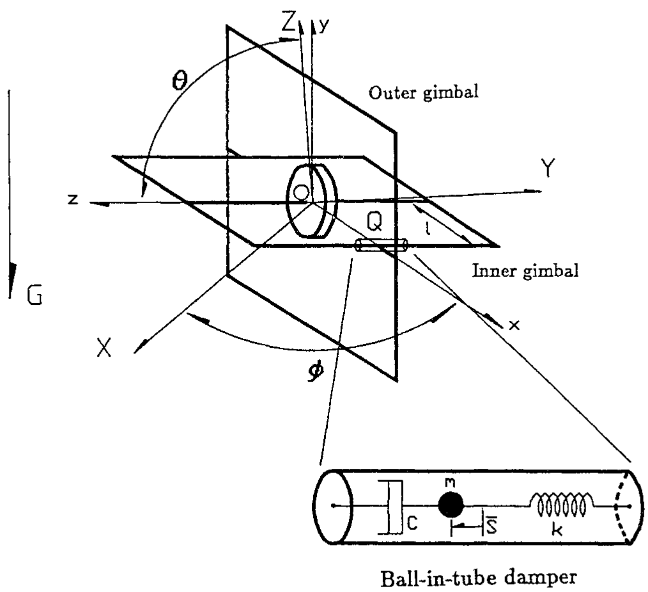

Fig. 1 Model of a two-degree-of-freedom gyroscope with nutation damper.

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proximate equations of motion on which the energy-sink method is based, are solved by employing the method of multiple scales. It is found that the decay time constant of the nutation angle based on the two-term-expansion solution can be reduced greatly if the natural frequency of the ball-in-tube damper is tuned to equal the nutation frequency of the gyroscope wth the damper removed. If the solutions are expanded in terms of the asymptotic sequence $\{\epsilon^0,\epsilon^1,\epsilon^2,\dots\}$, the decay time constant for the tuned gyroscope is valid only for the case in which the damping coefficient is large because it is found to be proportional to the damping coefficient and the unreasonable phenomenon that zero damping coefficient results in the minimum decay time constant comes out. Since, for the tuned gyroscope, the ratio of the magnitude of the kinetic energy of the ball to that of the rotor is of order $\epsilon^{1/2}$, this suggests that the solutions are better expanded in the new asymptotic sequence $\{\dots,\epsilon^{1/2},\epsilon^1,\epsilon^{3/2},\dots\}$ for the case in which the damping coefficient is small. The results show that the decay time constant is a concave function of the damping coefficient for the tuned gyroscope and is in good agreement with the results by integrating the exact equations of motion. The residual precession due to the gravity effect can be avoided if the damper installed in the inner gimbal is oriented perpendicular to the spin axis of the rotor.

Gyroscope Model

Let the case of the gyroscope be fixed in space and the X,Y,Z system fixed on the case be an inertial reference frame. The direction of gravity is along the negative direction of the Z axis. The outer gimbal can rotate freely about the Z axis and the amount of counterclockwise rotation is denoted by ϕ. The inner gimbal has one rotational degree of freedom with respect to the outer gimbal, i.e., the rotation of the xz plane about the x axis. This degree of freedom is represented by the variable θ. The gyroscope is designated to operate normally at the point where the plane of the inner gimbal is perpendicular to the plane of the outer gimbal, i.e., $\theta=90$ deg, as shown in Fig. 1, when the gyroscope is free of torques. The rotor is constrained to spin at a constant speed Ω. The ball-in-tube damper is considered first to be allocated in the inner gimbal and at such a position that the ball is at the point Q, which lies on the x axis, when the spring is undeformed. The system of inner gimbal and damper has been so balanced that the composite center of mass is located at the point O, which is also the mass center of the rotor and the origin of the x,y,z system. In the following, we let m, k, and c denote the mass, spring constant, and damping coefficient of the ball-in-tube damper, respectively; $\bar{s}$ is the displacement of the ball from the point Q; C and A the spin and transverse moments of inertial of the rotor with respect to the x,y,z system; A', B', and C' the principal moments of inertia of the inner gimbal with respect to the x,y,z system; and C'' the moment of inertia of the outer gimbal about the Z axis.

Problem Formulation

Equations of Motion

Let $\boldsymbol{\omega}=\omega_x\boldsymbol{i}+\omega_y\boldsymbol{j}+\omega_z\boldsymbol{k}$ denote the angular velocity of the inner gimbal and $\boldsymbol{\Omega}=\Omega_x\boldsymbol{i}+\Omega_y\boldsymbol{j}+\Omega_z\boldsymbol{k}$ the angular velocity of the rotor, where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the unit vectors of the x,y,z axes. Then, $(\omega_x,\omega_y,\omega_z)=(\dot\theta,\ \dot\phi\sin\theta,\ \dot\phi\cos\theta)$ and $(\Omega_x,\Omega_y,\Omega_z)=(\dot\theta,\ \dot\phi\sin\theta,\ \dot\phi\cos\theta+\Omega)$, where a dot denotes the differentiation with respect to time. The kinetic energy K_g of the inner gimbal, outer gimbal, and rotor is

$$K_g = \tfrac{1}{2}\Big[(A+A')\dot\theta^2+(A+B')\dot\phi^2\sin^2\theta+C(\dot\phi\cos\theta+\Omega)^2 + C'\dot\phi^2\cos^2\theta+C''\dot\phi^2\Big]$$

Let l denote the distance between point O and point Q and $\boldsymbol{r}(=l\boldsymbol{i}+\bar{s}\boldsymbol{k})$ the position vector of the ball. Then, the kinetic energy K_m of the ball is

$$K_m = \frac{m}{2}\left[\bar{s}^2\dot\phi^2\sin^2\theta+(l\dot\phi\cos\theta-\bar{s}\dot\theta)^2+(\dot{\bar{s}}-l\dot\phi\sin\theta)^2\right]$$

Since the kinetic energy K_{-m} of the ball when it is located at point Q is included in the kinetic energy of the inner gimbal, it must be deducted from the sum of K_g+K_m. The value of K_{-m} is

$$K_{-m} = \frac{m}{2}\left[(l\dot\phi\cos\theta)^2+(l\dot\phi\sin\theta)^2\right]$$

The potential energy of the gyroscopic system is $V=\tfrac{1}{2}k\bar{s}^2+mg\bar{s}\cos\theta$, where g is the gravitational acceleration. The Lagrangian is, therefore, $L=K_g+K_m-K_{-m}-V$. The Rayleigh's dissipation function of the linear damper is $F=\tfrac{1}{2}c\dot{\bar{s}}^2$. In order to nondimensionalize the equations of motion, we introduce the following new variables: $s=\bar{s}/l$, $\tau=\Omega t$, $\omega_n=\sqrt{k/m}$, $\omega_g=\sqrt{g/l}$, ζ(the damping factor)$=c/2\omega_n m$. Let $\epsilon=ml^2/A$. ϵ is a small parameter because m is assumed to be small in comparison with the mass of the rotor. The Lagrange equations[9] of motion of the gyroscopic system in nondimensional form and in terms of ϵ are

$$(1+\epsilon s^2)\ddot\theta-\epsilon s\ddot\phi\cos\theta=\left(1-\frac{C}{A}+\epsilon s^2\right)\sin\theta\cos\theta\dot\phi^2 - \left(\frac{C}{A}\right)\dot\phi\sin\theta-2\epsilon\dot s\dot\theta+\epsilon\left(\frac{\omega_g}{\Omega}\right)^2 s\sin\theta \tag{1}$$

$$-\epsilon s\ddot\theta\cos\theta+\left[(1+\epsilon s^2)\sin^2\theta+\frac{C}{A}\cos^2\theta\right]\ddot\phi-\epsilon\ddot s\sin\theta = -\epsilon s\dot\theta\sin\theta-2\left(1-\frac{C}{A}+\epsilon s^2\right)\sin\theta\cos\theta\dot\theta\dot\phi + 2\epsilon\dot s\dot\theta\cos\theta-2\epsilon s\dot s\dot\phi\sin^2\theta+\left(\frac{C}{A}\right)\dot\theta\sin\theta \tag{2}$$

$$\epsilon\ddot s-\epsilon\ddot\phi\sin\theta=\epsilon s\dot\theta^2+\epsilon s\sin^2\theta\dot\phi^2-\epsilon\left(\frac{\omega_n}{\Omega}\right)^2 s - \epsilon 2\zeta\left(\frac{\omega_n}{\Omega}\right)\dot s-\epsilon\left(\frac{\omega_g}{\Omega}\right)^2\cos\theta \tag{3}$$

where the superscript $(\cdot)$ indicates the differentiation with respect to τ. The moments of inertia A', B', C', and C'' do not appear in Eqs. (1–3) because the effect of gimbal inertia is well known[10] and is not considered here. Also, the effect of gravity is not considered for the present section. By vanishing the velocities and accelerations of all of the dependent variables in Eqs. (1–3), the equilibrium points are found to be that $s=0$ and θ and ϕ are any constants. We treat only the normal operation point, i.e., $\theta=\pi/2$. If the perturbed motion about this equilibrium point is considered, the solutions of the equations of motion linearized in θ show that the coning motion still exists and θ, ϕ, and s are of harmonic form. This is because the linearized system has two zero eigenvalues. Since the asymptotic behavior of solutions of a nonlinear system (and, hence, its stability type) cannot be determined by the linearization if the linearized system has zero or purely imaginary eigenvalues,[11] we, therefore, have to solve the nonlinear equations of motion to see whether the damper has the desired effect in reducing the coning motion.

Perturbation Analysis

The method of multiple scales[12] is employed here to find approximate analytical solutions. Letting $T_0=\tau$, $T_1=\epsilon\tau$,

$T_2 = \epsilon^2 \tau, \ldots$, we seek the solutions of Eqs. (1-3), with the gravity effect removed, in the forms,

$$\theta = \pi/2 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \cdots \tag{4a}$$

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots \tag{4b}$$

$$s = \epsilon s_1 + \epsilon^2 s_2 + \cdots \tag{4c}$$

where θ_i, ϕ_i, and s_i $(i = 1, 2, 3, \ldots)$ are functions of T_0, T_1, $T_2, \ldots$. Substituting Eqs. (4) into Eqs. (1-3) and equating the coefficients of various powers of ϵ to zero, we have

Order ϵ:

$$\frac{\partial^2\theta_1}{\partial^2 T_0^2} + \frac{C}{A}\frac{\partial\phi_1}{\partial T_0} = 0 \tag{5a}$$

$$\frac{\partial^2\phi_1}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_1}{\partial T_0} = 0 \tag{5b}$$

Order ϵ^2:

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = -2\frac{\partial^2\theta_1}{\partial T_0 \partial T_1} - \frac{C}{A}\frac{\partial\phi_1}{\partial T_1} \tag{6a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \frac{\partial^2 s_1}{\partial T_0^2} - 2\frac{\partial^2\phi_1}{\partial T_0 \partial T_1} + \frac{C}{A}\frac{\partial\theta_1}{\partial T_1} \tag{6b}$$

$$\frac{\partial^2 s_1}{\partial T_0^2} + 2\zeta\left(\frac{\omega_n}{\Omega}\right)\frac{\partial s_1}{\partial T_0} + \left(\frac{\omega_n}{\Omega}\right)^2 s_1 - \frac{\partial^2\phi_1}{\partial T_0^2} = 0 \tag{6c}$$

Order ϵ^3

....

$$\frac{\partial^2 s_2}{\partial T_0^2} + 2\zeta\left(\frac{\omega_n}{\Omega}\right)\frac{\partial s_2}{\partial T_0} + \left(\frac{\omega_n}{\Omega}\right)^2 s_2 - \frac{\partial^2\phi_2}{\partial T_0^2}$$

$$= -2\frac{\partial^2 s_1}{\partial T_0 \partial T_1} + 2\frac{\partial^2\phi_1}{\partial T_0 \partial T_1} - 2\zeta\left(\frac{\omega_n}{\Omega}\right)\frac{\partial s_1}{\partial T_1} \tag{7}$$

Equations (5a), (5b), and (6c) must be solved simultaneously. Their solutions are assumed in the form as $(\theta_1, \phi_1, s_1)^T = (C_1, C_2, C_3)^T e^{\lambda T_0}$, where C_1, C_2, and C_3 are the components of eigenvectors. The substitution of the assumed solutions into Eqs. (5a), (5b), and (6c) leads to the following equation:

$$\begin{bmatrix} \lambda^2 & (C/A)\lambda & 0 \\ -(C/A)\lambda & \lambda^2 & 0 \\ 0 & -\lambda^2 & (\omega_n/\Omega)^2 + \lambda^2 + 2\zeta(\omega_n/\Omega)\lambda \end{bmatrix}$$

$$\times \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = 0 \tag{8}$$

Nontrivial solutions to Eq. (8) exist if and only if the determinant of the coefficient matrix is nonzero. This requirement leads to a characteristic equation for the eigenvalues, and its solutions (i.e., the eigenvalues) are

$$\lambda = 0, \quad 0, \quad \pm\frac{C}{A}i, \quad -\zeta\left(\frac{\omega_n}{\Omega}\right) \pm i\sqrt{1-\zeta^2}\left(\frac{\omega_n}{\Omega}\right) \tag{9}$$

The unbounded solution corresponding to the repeated eigenvalue 0 is $(C_1^2, C_2^2, C_3^2)^T T_0 e^{0T_0}$. Substituting it into Eqs. (5a), (5b), and (6c), we obtain $C_1^2 = C_2^2 = C_3^2 = 0$. The other eigenvectors are obtained by substituting eigenvalues from Eq. (9) into Eq. (8) and solving for the eigenvectors. The general solutions of θ_1, ϕ_1, and s_1 formed in this manner are

$$\theta_1 = iB_1(T_1)e^{i(C/A)T_0} + B_3(T_1) + \text{cc}$$

$$\phi_1 = B_1(T_1)e^{i(C/A)T_0} + B_4(T_1) + \text{cc}$$

$$s_1 = \frac{-(C/A)^2 B_1(T_1)}{\Sigma} e^{i(C/A)T_0}$$

$$+ B_2(T_1)e^{(-\zeta + i\sqrt{1-\zeta^2})(\omega_n/\Omega)T_0} + \text{cc} \tag{10}$$

where

$$\Sigma = \left(\frac{\omega_n}{\Omega}\right)^2 - \left(\frac{C}{A}\right)^2 + 2i\zeta\left(\frac{\omega_n}{\Omega}\right)\left(\frac{C}{A}\right)$$

The $B_i(T_1)(i = 1, \ldots, 4)$ are undetermined at this level of approximation; they are determined at the next level of approximation by eliminating the secular terms. The notation cc stands for the complex conjugate of the preceding terms. The substitution of Eqs. (10) into Eqs. (6a), (6b), and (7) yields

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = \frac{C}{A}\frac{dB_1}{dT_1}e^{i(C/A)T_0} - \frac{C}{A}\frac{dB_4}{dT_1} + \text{cc} \tag{11a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \left[\frac{(C/A)^4 B_1}{\Sigma} - i\frac{C}{A}\frac{dB_1}{dT_1}\right]e^{i(C/A)T_0}$$

$$+ \frac{C}{A}\frac{dB_3}{dT_1} + (\Gamma_1)^2\left(\frac{\omega_n}{\Omega}\right)^2 B_2 e^{\Gamma_1(\omega_n/\Omega)T_0} + \text{cc} \tag{11b}$$

$$\frac{\partial^2 s_2}{\partial T_0^2} + 2\zeta\left(\frac{\omega_n}{\Omega}\right)\frac{\partial s_2}{\partial T_0} + \left(\frac{\omega_n}{\Omega}\right)^2 s_2 - \frac{\partial^2\phi_2}{\partial T_0^2}$$

$$= \frac{2i(C/A)\left[(\omega_n/\Omega)^2 + i\zeta(\omega_n/\Omega)(C/A)\right]}{\Sigma}\frac{dB_1}{dT_1}e^{i(C/A)T_0}$$

$$-2i\sqrt{1-\zeta^2}\frac{\omega_n}{\Omega}\frac{dB_2}{dT_1}e^{\Gamma_1(\omega_n/\Omega)T_0} + \text{cc} \tag{11c}$$

where

$$\Gamma_1 = -\zeta + i\sqrt{1-\zeta^2} \tag{12}$$

Equations (11a) and (11b) do not contain the dependent variables s, and so they are solved separately from Eq. (11c). The inhomogeneous terms of Eqs. (11a) and (11b) are divided into three classes, i.e., 1) the terms including $e^{i(C/A)T_0}$, 2) the terms including $e^{(-\zeta + i\sqrt{1-\zeta^2})(\omega_n/\Omega)T_0}$, and 3) the nonexponential terms. According to these, Eqs. (11a) and (11b) are divided into three sets and their solutions are the sum of the each solution of the three sets of equations. The equations of the first set are

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = \frac{C}{A}\frac{dB_1}{dT_1}e^{i(C/A)T_0} + \text{cc} \tag{13a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \left[\frac{(C/A)^4 B_1}{\Sigma} - i\frac{C}{A}\frac{dB_1}{dT_1}\right]e^{i(C/A)T_0} + \text{cc} \tag{13b}$$

Since the homogeneous parts of Eqs. (13a) and (13b) have solutions proportional to $\exp[i(C/A)T_0]$, the particular solutions of Eqs. (13a) and (13b) will have secular terms. Therefore, we seek a particular solution free of secular terms in the form[13]

$$\theta_2 = P(T_1)e^{i(C/A)T_0} + \text{cc} \tag{14a}$$

$$\phi_2 = Q(T_1)e^{i(C/A)T_0} + \text{cc} \tag{14b}$$

Substituting Eqs. (14) into Eqs. (13) and equating the coefficient of $\exp[i(C/A)T_0]$, we obtain

$$\begin{bmatrix} -\left(\frac{C}{A}\right)^2 & i\left(\frac{C}{A}\right)^2 \\ -i\left(\frac{C}{A}\right)^2 & -\left(\frac{C}{A}\right)^2 \end{bmatrix} \begin{Bmatrix} P \\ Q \end{Bmatrix} = \begin{Bmatrix} \frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1} \\ \frac{(C/A)^4}{\Sigma}B_1 - i\frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1} \end{Bmatrix} \tag{15}$$

Since the determinant of the coefficient of Eq. (15) is zero, the solvability condition is

$$\frac{\mathrm{d}B_1}{\mathrm{d}T_1} = -i\left[\frac{(C/A)^3}{\Sigma}B_1 - i\frac{\mathrm{d}B_1}{\mathrm{d}T_1}\right] \tag{16}$$

Integrating Eq. (16), we obtain

$$B_1(T_1) = b_1\exp\left[\frac{-i(C/A)^3}{2\Sigma}T_1\right] \tag{17}$$

where the constant b_1 is determined from the initial conditions. By substituting Eq. (17) into Eq. (15) and solving Eq. (15) we obtain

$$Q = \frac{(C/A)^2}{2\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1}$$

$$P = \frac{i(C/A)^2}{\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1}$$

Hence, the particular solutions of Eqs. (13) are

$$\theta_2^1 = \frac{i(C/A)^2}{\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1}e^{i(C/A)T_0} + \mathrm{cc} \tag{18a}$$

$$\theta_2^1 = \frac{(C/A)^2}{2\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1}e^{i(C/A)T_0} + \mathrm{cc} \tag{18b}$$

The equations of the second set are

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = -\frac{C}{A}\frac{\mathrm{d}B_4}{\mathrm{d}T_1} + \mathrm{cc} \tag{19a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \frac{C}{A}\frac{\mathrm{d}B_3}{\mathrm{d}T_1} + \mathrm{cc} \tag{19b}$$

Since the homogeneous parts of Eqs. (19) have the solution proportional to $\exp[i0T_0]$, for the existence of a nontrivial solution the solvability condition is

$$\frac{\mathrm{d}B_4}{\mathrm{d}T_1} = \frac{\mathrm{d}B_3}{\mathrm{d}T_1} = 0 \tag{20}$$

Hence,

$$B_3(T_1) = b_3 \tag{21a}$$

$$B_4(T_1) = b_4 \tag{21b}$$

where constants b_3 and b_4 are determined from the initial conditions. The particular solutions θ_2^2 and ϕ_2^2 of Eqs. (19) are

$$\theta_2^2 = \phi_2^2 = 0 \tag{22}$$

The equations of the third set are

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = 0 \tag{23a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \Gamma_1\left(\frac{\omega_n}{\Omega}\right)^2 B_2 e^{\Gamma_1(\omega_n/\Omega)T_0} + \mathrm{cc} \tag{23b}$$

The inhomogeneous parts have no frequency identical to those of the homogeneous part; the particular solutions θ_2^3 and ϕ_2^3 of Eqs. (23) are solved in terms of $B_1(T_1)$. By superposition the solutions of Eqs. (11a) and (11b) are

$$\theta_2 = \frac{i(C/A)^2}{\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1 + i(C/A)T_0} - \frac{(C/A)\Gamma_1(\omega_n/\Omega)}{(\Gamma_1)^2(\omega_n/\Omega)^2 + (C/A)^2}B_2 e^{\Gamma_1(\omega_n/\Omega)T_0} + \mathrm{cc} \tag{24a}$$

$$\phi_2 = \frac{(C/A)^2}{2\Sigma}b_1 e^{-\frac{i(C/A)^3}{2\Sigma}T_1 + i(C/A)T_0} + \frac{(\Gamma_1)^2(\omega_n/\Omega)^2}{(\Gamma_1)^2(\omega_n/\Omega)^2 + (C/A)^2}B_2 e^{\Gamma_1(\omega_n/\Omega)T_0} + \mathrm{cc} \tag{24b}$$

The substitution of Eqs. (17) and (24b) into Eq. (11c) yields

$$\frac{\partial^2 s_2}{\partial T_0^2} + 2\zeta\frac{\omega_n}{\Omega}\frac{\partial s_2}{\partial T_0} + \left(\frac{\omega_n}{\Omega}\right)^2 s_2 = \frac{(C/A)^4\left[(\omega_n/\Omega)^2 + (C/A)^2\right]}{2\Sigma^2} \times b_1\exp\left\{-i\frac{(C/A)^3}{2\Sigma}T_1 + i(C/A)T_0\right\} + \left[\frac{(\Gamma_1)^4(\omega_n/\Omega)^4}{(\Gamma_1)^2(\omega_n/\Omega)^2 + (C/A)^2}B_2 - 2i\sqrt{1-\zeta^2}\frac{\omega_n}{\Omega}\frac{\mathrm{d}B_2}{\mathrm{d}T_1}\right] \times \exp\left[\Gamma_1\frac{\omega_n}{\Omega}T_0\right] + \mathrm{cc} \tag{25}$$

In order to eliminate the secular terms, we must make the coefficient of the second exponential term on the right side of Eq. (25) zero, that is,

$$2i\sqrt{1-\zeta^2}\frac{\mathrm{d}B_2}{\mathrm{d}T_1} = \frac{(\Gamma_1)^4(\omega_n/\Omega)^4}{(\Gamma_1)^2(\omega_n/\Omega)^2 + (C/A)^2}B_2 \tag{26}$$

By integrating Eq. (26), we have

$$B_2(T_1) = b_2\exp\left\{\frac{-i(\Gamma_1)^4(\omega_n/\Omega)^3 T_1}{2\sqrt{1-\zeta^2}(\omega_n/\Omega)^2 + (C/A)^2}\right\} \tag{27}$$

where the constant b_2 is determined from initial conditions. Then, the solution s_2 of Eq. (25) is

$$s_2 = \frac{(C/A)^4[\Gamma_2]}{2\Sigma^3}b_1 e^{\left|-i\frac{(C/A)^3}{2\Sigma}T_1 + i(C/A)T_0\right|} + \mathrm{cc} \tag{28}$$

where $\Gamma_2 = (\omega_n/\Omega)^2 + (C/A)^2$. By substituting B_1, B_2, B_3, and B_4 into Eqs. (10) and simplifying it, we find

$$\theta_1 = \exp\left\{-\frac{\zeta(\omega_n/\Omega)(C/A)^4}{|\Sigma|^2}T_1\right\} \times \left[b_1\exp\left\{i\left(\left(\frac{C}{A}\right)\left[T_0 - \frac{(C/A)^2\Gamma_3}{2|\Sigma|^2}T_1\right] + \frac{\pi}{2}\right)\right\}\right] + b_3 + \mathrm{cc} \tag{29a}$$

$$\phi_1 = \exp\left\{-\frac{\zeta(\omega_n/\Omega)(C/A)^4}{|\Sigma|^2}T_1\right\} \times \left[b_1\exp\left\{i\frac{C}{A}\left[T_0 - \frac{(C/A)^2\Gamma_3}{2|\Sigma|^2}T_1\right]\right\}\right] + b_4 + \mathrm{cc} \tag{29b}$$

and

$$
\begin{aligned}
s_1 = & -\frac{(C/A)^2}{|\Sigma|} \exp\left\{-\frac{\zeta(\omega_n/\Omega)(C/A)^4}{|\Sigma|^2} T_1\right\} \\
& \times \left[b_1 \exp\left\{ i\left(\frac{C}{A}\left[T_0 - \frac{(C/A)^2(\Gamma_3)}{2|\Sigma|^2} T_1\right]\right.\right.\right. \\
& \left.\left.\left. - \arctan\frac{2\zeta(\omega_n/\Omega)(C/A)}{\Gamma_3}\right)\right\}\right] \\
& + \exp\left\{-\zeta\frac{\omega_n}{\Omega}\left(T_0 + \frac{2(2\zeta^2-1)(C/A)^2(\omega_n/\Omega)^2}{|\Sigma|^2} T_1\right)\right\} \\
& \times \left[b_2 \exp\left\{ i\left(\sqrt{1-\zeta^2}\frac{\omega_n}{\Omega} T_0\right.\right.\right. \\
& \left.\left.\left. - \frac{(8\zeta^4 - 8\zeta^2 + 1)(C/A)^2 + (2\zeta^2-1)(\omega_n/\Omega)^2}{2\sqrt{1-\zeta^2}|\Sigma|^2}\left(\frac{\omega_n}{\Omega}\right)^3 T_1\right)\right\}\right] \\
& + \mathrm{cc}
\end{aligned}
\tag{29c}
$$

where

$$
\Gamma_3 = \left(\frac{\omega_n}{\Omega}\right)^2 - \left(\frac{C}{A}\right)^2
$$

Decay Time Constant

The expansion of θ, ϕ, and s is up to $\mathcal{O}(\epsilon^3)$. Since each term except the constant term of the solutions θ_1, θ_2, ϕ_1, ϕ_2, s_1, and s_2 is the product of the exponentially decreasing function by the harmonic function, the coning motion (nutation and precession) is really damped. If the sequence $\{\theta_1, \theta_2, \ldots\}$ is convergent, the magnitude of $\epsilon^2\theta_2$ is much less than that of $\epsilon\theta_1$, and the decaying behavior of θ is therefore determined by the θ_1. The decay time constant τ_c is defined as the time needed for θ to decay to equal the value $\exp(-1)$. Then, from Eqs. (29a) and (29b), τ_c is

$$
\tau_c = \frac{\left[(\omega_n/\Omega)^2 - (C/A)^2\right]^2 + 4\zeta^2(\omega_n/\Omega)^2(C/A)^2}{\epsilon\zeta(\omega_n/\Omega)(C/A)^4} \tag{30}
$$

The decay time constant is expected to be as small as possible, and so the choice of the parameters of the damper is of importance. We consider the effect of the damping factor ζ on τ_c with other parameters fixed. The first and second derivatives of τ_c with respect to ζ are

$$
\frac{\partial\tau_c}{\partial\zeta} = \frac{1}{\epsilon(\omega_n/\Omega)(C/A)^4} \times \frac{4\zeta^2(\omega_n/\Omega)^2(C/A)^2 - \left[(\omega_n/\Omega)^2 - (C/A)^2\right]^2}{\zeta^2} \tag{31}
$$

$$
\frac{\partial^2\tau_c}{\partial\zeta^2} = \frac{1}{\epsilon(\omega_n/\Omega)(C/A)^4}\,\frac{2\left[(\omega_n/\Omega)^2 - (C/A)^2\right]^2}{\zeta^3} \tag{32}
$$

Equation (32) shows that $\partial^2\tau_c/\partial\zeta^2$ is always positive. Therefore, the value of ζ obtained by setting $\partial\tau_c/\partial\zeta$ to equal zero will minimize τ_c. This also tells us that τ_c is a concave function of ζ and that the damping coefficient cannot be made to small or too large in order to have better damping effect on the elimination of the coning motion. From Eqs. (30) and (31), the minimum value of τ_c is

$$
(\tau_c)_{\min} = \frac{4|(\omega_n/\Omega)^2 - (C/A)^2|}{\epsilon(C/A)^3} \tag{33}
$$

which occurs at

$$
\zeta = \frac{|(\omega_n/\Omega)^2 - (C/A)^2|}{2(\omega_n/\Omega)(C/A)} \tag{34}
$$

Equations (33) and (34) are valid only for the nonresonant case in which the undamped natural frequency ω_n of the ball-in-tube damper is not tuned to equal the nutation frequency $(C/A)\Omega$ of the gyroscope with the damper removed,[14] otherwise the value of $(\tau_c)_{\min}$ is zero and also $\zeta = 0$. The closer to $(C/A)\Omega$ the ω_n is, however, the smaller is the τ_c. Regarding the resonant case, let us look at the solution s_1. The comparison of the first term of Eq. (29c) with Eq. (29a) shows that the harmonic motion of the ball inside the damper has the same frequency $[(C/A) - \epsilon(C/A)^2\Gamma_3/(2|\Sigma|^2)]$ as the vibrating gimbal, but has a different phase. If m and k are chosen properly so that the ω_n is equal to $(C/A)\Omega$, Γ_3 will be equal to zero and the first part of s_1 will be in phase with ϕ_1 and $\pi/2$ out of phase with θ_1. Also, their common frequency will become exactly the nutation frequency $(C/A)\Omega$. In this resonant case, the decay time constant [from Eq. (30)] is

$$
\tau_c = \frac{4\zeta(\omega_n/\Omega)}{\epsilon(C/A)^2} = \frac{4\zeta}{\epsilon}\frac{\Omega}{\omega_n} = \frac{4\zeta}{\epsilon}\left(\frac{A}{C}\right) \tag{35}
$$

Equation (35) shows that τ_c is proportional to ζ. The decay time constant given by Eq. (35) and the approach in this section are valid only for the case in which the damping factor ζ of a tuned gyroscope [which means that $\omega_n = (C/A)\Omega$] is large, otherwise τ_c can be made zero by setting $\zeta = 0$. The reason why the method of multiple scales (MMS) fails to give the correct time constant for the resonant and $\zeta \to 0$ case is that the MMS fails to remove the secular terms and results in the inability to produce a uniformly valid solution. Consider the $\omega_n = (C/A)\Omega$ and $\zeta = 0$ case. In such case, the solution for s_1 in Eqs. (10) is not valid because of the zero denominator (i.e., $\Sigma = 0$) in the s_1 equation. For this case, Eq. (6c) reduces to

$$
\frac{\partial^2 s_1}{\partial T_0^2} + \left(\frac{C}{A}\right)^2 s_1 = \frac{\partial^2\phi_1}{\partial T_0^2} \tag{36a}
$$

The alternate solution for θ_1 and ϕ_1 is

$$
\theta_1 = a_1(T_1)e^{i(C/A)T_0}, \quad \phi_1 = -ia_1(T_1)e^{i(C/A)T_0} + a_2(T_1) \tag{36b}
$$

Substituting Eq. (36b) into Eq. (36a), we obtain

$$
\frac{\partial^2 s_1}{\partial T_0^2} + \left(\frac{C}{A}\right)^2 s_1 = ia_1(T_1)\left(\frac{C}{A}\right)^2 e^{i(C/A)T_0} \tag{36c}
$$

The a_1 must be zero if the secular term on the right side of Eq. (36c) is to be eliminated. This will lead to the incorrect result that the angles θ_1 and ϕ_1 do not oscillate at the resonant (nutation) frequency. Therefore, the MMS with the asymptotic sequence $\{\epsilon^0, \epsilon^1\epsilon^2, \ldots\}$ fails to give a valid solution. Therefore, another more appropriate asymptotic sequence must be sought for the case where the gyroscope is tuned and the damping factor is small.

An interesting observation is found that the time constant, Eq. (30), is in the same form as for the single-spin satellite with a spring-dashpot-mass damper on the spinning body [Eq. (41) of Ref. 15)]. Since Ref. 15 obtained the time constant by employing the energy-sink approximation, the solution for the time constant by the energy-sink method, therefore, is the same to first order in ϵ for the nonresonant case as that by the method of multiple scales. As to the resonant case, Ref. 15 failed to explain the unreasonable phenomenon that the time constant approaches zero as the damping factor tends to zero.

Resonant Case

Based on the estimation of the ratio of the kinetic energy of the damper mass (the ball) to that of the rotor for the resonant case a suitable asymptotic sequence may be established on physical grounds. Denote s_a as the maximum magnitude of the displacement of the ball, then in the resonant case the maximum kinetic energy of the ball is $\frac{1}{2}m\omega_n^2(s_a)^2$. Since the solutions for nutation and precession of the gyroscope without ball-in-tube damper are

$$\theta = c_0 \sin\left(\frac{C}{A}\Omega t + \beta_1\right) \tag{37a}$$

$$\phi = c_0 \cos\left(\frac{C}{A}\Omega t + \beta_2\right) \tag{37b}$$

where c_0, β_1, and β_2 are constant. The kinetic energy K_c of the coning motion of the rotor is $\frac{1}{2}(A\omega_x^2 + A\omega_y^2) = \frac{1}{2}(A\dot{\theta}^2 + A\dot{\phi}^2 \sin^2\theta) \approx \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2)$ for small θ. Denote θ_a as the maximum magnitude of θ, then the maximum values of $\dot{\theta}$ and $\dot{\phi}$ are $(C/A)\Omega\theta_a$ and $(K_c)_{\max} = \frac{1}{2}A(C/A)^2\Omega^2(\theta_a)^2$. Assume that the energy of coning motion of the rotor is transferred completely to the ball, i.e.,

$$\frac{1}{2}A\left(\frac{C}{A}\right)^2\Omega^2\theta_a^2 = \frac{1}{2}m\omega_n^2 s_a^2 \tag{38}$$

Since $\omega_n/\Omega = C/A$ for the resonant case, Eq. (38) becomes

$$\left(\frac{\theta_a}{s_a}\right)^2 = \frac{m}{A} \tag{39}$$

To nondimensionalize s_a, we denote $\bar{s}_a = s_a/l$, then Eq. (39) becomes

$$\left(\frac{\theta_a}{\bar{s}_a}\right)^2 = \frac{ml^2}{A} = \epsilon$$

So, for the resonant case, if the magnitude of displacement of the ball is of order 1, the magnitude of θ is of order $\epsilon^{1/2}$. Hence, in the perturbation analysis, the term $\theta_i \epsilon^{i/2}$ (i is an integer) in the expanded solution for θ must correspond to the term $s_i \epsilon^{(i-1)/2}$ in the expanded solution for s.

Perturbation Analysis

Denote $T_0 = \tau$, $T_1 = \epsilon^{1/2}\tau$, and $T_2 = \epsilon\tau$. Equations (4) are now modified as

$$\theta = \pi/2 + \epsilon^{1/2}\theta_1 + \epsilon^1\theta_2 + \epsilon^{3/2}\theta_3 + \cdots \tag{40a}$$

$$\phi = \epsilon^{1/2}\phi_1 + \epsilon^1\phi_2 + \epsilon^{3/2}\phi_3 + \cdots \tag{40b}$$

$$s = s_0 + \epsilon^{1/2}s_1 + \epsilon^1 s_2 + \cdots \tag{40c}$$

In order to take account of the effect of small damping factor we define

$$\zeta = \epsilon^{1/2}\eta \tag{41}$$

The governing equations for the first-order solutions θ_1, ϕ_1, s_0 and second-order solutions θ_2, ϕ_2, s_1 are obtained by substituting Eqs. (40) and (41) into Eqs. (1-3) and equating the coefficients of $\epsilon^{1/2}$, ϵ^1, and $\epsilon^{3/2}$ to zero. The first-order solutions are

$$\theta_1 = iB_1(T_1)e^{i(C/A)T_0} + B_3(T_1) + \text{cc} \tag{42a}$$

$$\phi_1 = B_1(T_1)e^{i(C/A)T_0} + B_4(T_1) + \text{cc} \tag{42b}$$

$$s_0 = B_2(T_1)e^{i(C/A)T_0} + \text{cc} \tag{42c}$$

The governing equations for θ_2, ϕ_2 are divided into two sets according to the two different exponential terms of their inhomogeneous parts. Accordingly, the equations of the first set are

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = \frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1}e^{i(C/A)T_0} + \text{cc} \tag{43a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \left[-i\frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1} - \left(\frac{C}{A}\right)^2 B_2\right]e^{i(C/A)T_0} + \text{cc} \tag{43b}$$

The equations of the second set are

$$\frac{\partial^2\theta_2}{\partial T_0^2} + \frac{C}{A}\frac{\partial\phi_2}{\partial T_0} = -\frac{C}{A}\frac{\mathrm{d}B_4}{\mathrm{d}T_1} + \text{cc} \tag{44a}$$

$$\frac{\partial^2\phi_2}{\partial T_0^2} - \frac{C}{A}\frac{\partial\theta_2}{\partial T_0} = \frac{C}{A}\frac{\mathrm{d}B_3}{\mathrm{d}T_1} + \text{cc} \tag{44b}$$

The equation governing s_1 is

$$\frac{\partial^2 s_1}{\partial T_0^2} + \left(\frac{\omega_n}{\Omega}\right)^2 s_1 = -2i\frac{C}{A}\frac{\mathrm{d}B_2}{\mathrm{d}T_1}e^{i(C/A)T_0} - \left(\frac{C}{A}\right)^2 B_1 e^{i(C/A)T_0} - 2\eta i\left(\frac{C}{A}\right)^2 B_2 e^{i(C/A)T_0} + \text{cc} \tag{45}$$

The solvability condition needed for eliminating the secular term on the right side of Eqs. (43) is

$$\begin{vmatrix} -\left(\frac{C}{A}\right)^2 & \frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1} \\ -i\left(\frac{C}{A}\right)^2 & -i\frac{C}{A}\frac{\mathrm{d}B_1}{\mathrm{d}T_1} - \left(\frac{C}{A}\right)^2 B_2 \end{vmatrix} = 0 \tag{46}$$

Expanding Eq. (46), we have

$$\frac{\mathrm{d}B_1}{\mathrm{d}T_1} = \frac{i}{2}\frac{C}{A}B_2 \tag{47}$$

The solvability condition for Eqs. (44) is

$$\frac{\mathrm{d}B_4}{\mathrm{d}T_1} = \frac{\mathrm{d}B_3}{\mathrm{d}T_1} = 0 \tag{48}$$

The integration of Eq. (48) yields $B_3(T_1) = b_3$ and $B_4(T_1) = b_4$. To eliminate the secular term in the solution of Eq. (45), the coefficient of $\exp[i(C/A)T_0]$ in the inhomogeneous part of Eq. (45) must be zero, i.e.,

$$2\frac{\mathrm{d}B_2}{\mathrm{d}T_1} + 2\eta\frac{C}{A}B_2 - i\frac{C}{A}B_1 = 0 \tag{49}$$

Differentiating Eq. (49) with respect to T_1 and using Eq. (47), we obtain

$$2\frac{\mathrm{d}^2B_2}{\mathrm{d}T_1^2} + 2\eta\frac{C}{A}\frac{\mathrm{d}B_2}{\mathrm{d}T_1} + \frac{1}{2}\left(\frac{C}{A}\right)^2 B_2 = 0 \tag{50}$$

The solution of Eq. (50) depends on the value of η. If $\eta < 1$, which is the underdamped case, the solution B_2 is

$$B_2(T_1) = b_2 \exp\left[\frac{1}{2}\Re\frac{C}{A}T_1\right] + b_2' \exp\left[\frac{1}{2}\bar{\Re}\frac{C}{A}T_1\right] \tag{51a}$$

where

$$\Re = -\eta + i\sqrt{1-\eta^2}, \qquad \bar{\Re} = -\eta - i\sqrt{1-\eta^2}$$

If $\eta = 1$, which is referred to as critical damping, the solution B_2 is

$$B_2(T_1) = (b_2 + b_2' T_1)\exp\left[-\frac{1}{2}\frac{C}{A}T_1\right] \tag{51b}$$

If $\eta > 1$, which is the overdamped case, the solution B_2 is

$$B_2(T_1) = b_2 \exp\left[\frac{1}{2}\left(-\eta + \sqrt{\eta^2-1}\right)\frac{C}{A}T_1\right] + b_2' \exp\left[\frac{1}{2}\left(-\eta - \sqrt{\eta^2-1}\right)\frac{C}{A}T_1\right] \tag{51c}$$

The integration constants b_2, b_2', b_3, and b_4 are determined from the initial conditions. B_1 is obtained by substituting B_2 into Eq. (47) and integrating it. Substituting B_1, B_2, B_3, and B_4 into Eqs. (42), we have, when $\eta < 1$,

$$\theta_1 = \frac{-b_2}{\Re} \exp\left[\frac{1}{2}\Re\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + \frac{-b_2'}{\bar{\Re}}\exp\left[\frac{1}{2}\bar{\Re}\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + b_3 + \text{cc} \tag{52a}$$

$$\phi_1 = \frac{ib_2}{\Re}\exp\left[\frac{1}{2}\Re\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + \frac{ib_2'}{\bar{\Re}}\exp\left[\frac{1}{2}\bar{\Re}\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + b_4 + \text{cc} \tag{52b}$$

and

$$s_0 = b_2\exp\left[\frac{1}{2}\Re\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + b_2'\exp\left[\frac{1}{2}\bar{\Re}\frac{C}{A}T_1\right]\exp\left[i\frac{C}{A}T_0\right] + \text{cc} \tag{52c}$$

The solutions of θ_1, ϕ_1, and s_0 for the other two case, i.e., $\eta = 1$ and $\eta > 1$, are of similar form.[16]

Decay Time Constant

Based on the solutions of θ_1, ϕ_1, and s_0, the approximate decay time constant is,

For $\eta < 1$:

$$\tau_c = \frac{2}{\epsilon^{1/2}\eta}\frac{A}{C} \tag{53}$$

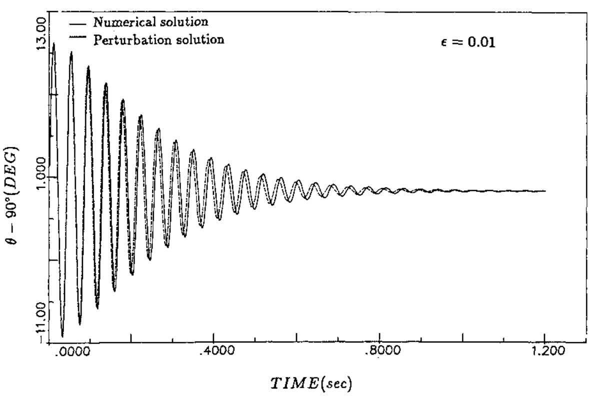

Fig. 2 Time history of the nutation angle; $\dot{\theta}(0) = 1600$ deg/s.

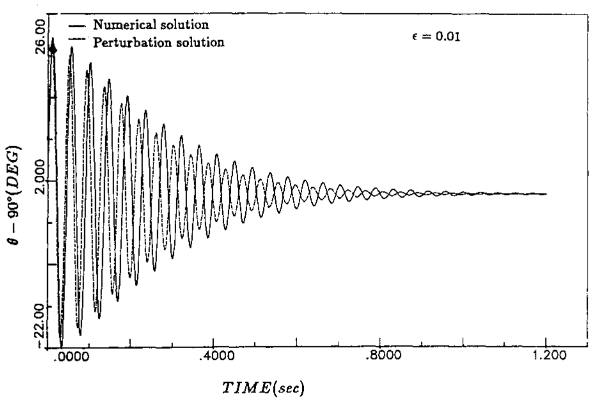

Fig. 3 Time history of the nutation angle; $\dot{\theta}(0) = 3200$ deg/s.

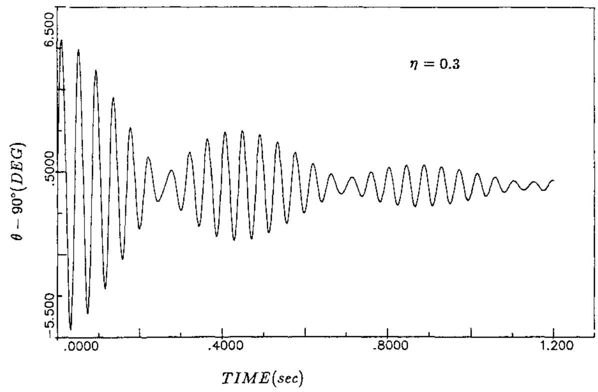

Fig. 4 Time history of the nutation angle for the underdamped case.

For $\eta = 1$:

$$\tau_c = \frac{2}{\epsilon^{1/2}}\frac{A}{C} \tag{54}$$

For $\eta > 1$:

$$\tau_c = \frac{2}{\epsilon^{1/2}\left(\eta - \sqrt{\eta^2 - 1}\right)}\frac{A}{C} \tag{55}$$

It is obvious that the minimum of τ_c occurs at $\eta = 1$ if τ_c is considered as a function of η. If the damping factor is large, i.e., $\eta \gg 1$, Eq. (55) is approximated to

$$\tau_c = \frac{2}{\epsilon^{1/2}\left\{\eta - \eta\left[1 - (1/2\eta^2)\right]\right\}}\frac{A}{C} \approx \frac{4\eta}{\epsilon^{1/2}}\frac{A}{C} = \frac{4\zeta}{\epsilon}\frac{A}{C} \tag{56}$$

Equation (56) is exactly the same as Eq. (35). This indicates again that the decay time constant from the solutions expanded in the sequence $\{\epsilon, \epsilon^2, \epsilon^3, \ldots\}$ is valid only for large damping factor.

Numerical Results and Discussions

To establish the validity of the approximate perturbation solutions and test what degree of nonlinearity for θ and ϕ the perturbation solutions are applicable to, the solutions given by Eqs. (4) and (40) are compared with the solutions by numerically integrating the exact equations [Eqs. (1-3)] of motion. We consider the special example for which $\epsilon = 0.01$, $C/A = 1.5$, $\Omega = 1.0 \times 10^2$ rad/s, $\omega_n = 1.5 \times 10^2$ rad/s, and the initial conditions are $\theta(0) = 90$ deg, $\phi(0) = 0$ deg, $s(0) = 0$, $\dot{\phi}(0) = 0$ deg/s, $\dot{s}(0) = 0$/s. Figures 2 and 3 are the comparison of the time history of the nutation angle from the first-order perturbation solution [i.e., Eq. (40)] with that from the numerical solution for the critical damping case with different initial values of $\dot{\theta}(0)$. These two solutions, as shown in Fig. 2, match well when $\dot{\theta}(0) < 1600$ deg/s and the maximum value of the $\theta(t)$ is about 13 deg. When $\dot{\theta}(0) > 3200$ deg/s, the perturbation solution deviates from the numerical solution apparently as shown in Fig. 3 where the maximum value of θ is about 23 deg. In spite of this deviation, the order of the decay rate of the nutation angle for these two solutions is very close. Therefore, the first-order perturbation solutions of Eqs. (40) are reliable.

Figure 4 is the time history of the nutation angle from the numerical solution for the underdamped case with $\dot{\theta}(0) = 800$ deg/s. The decay time constant τ_c as a function of η with the other parameters fixed is plotted in Fig. 5. Here, the τ_c of the numerical solution is determined by using the logarithmic decrement method. Figure 5 shows that the numerical solution is almost identical to the perturbation solution given by Eqs. (53-55), and for the case where $\eta \gg 1$ the τ_c from Eq. (35) approaches that from Eq. (55). Since Fig. 5 reveals that τ_c is a concave function of η, this implies that the damp-

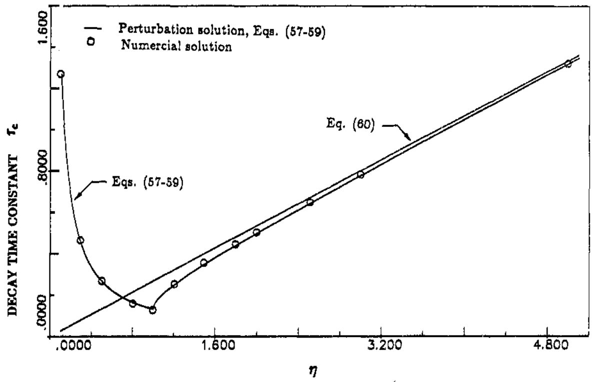

Fig. 5 Decay time constant τ_c vs η.

ing factor cannot be designed too large or too small. From $\zeta = c/2\omega_n m$ and using $\omega_n = (C/A)\Omega$, the value of the optimal damping coefficient occurring at $\eta = 1$ is

$$c = 2m\Omega \frac{C}{A}\sqrt{\frac{ml^2}{A}} \tag{57}$$

Gravity Effects

The gravitational force alters the motion of the damper mass, which in turn affects the motion of the rotor and results in a change in the decay time constant. Alfriend[15] analyzed the effect of gravity on the time constant for the satellite with a spring-mass-dashpot damper. The equation for the time constant here derived which is similar to Eq. (30), is applicable to the case for which the gyroscope is tuned and the damping factor is not small. Here, we consider the gravity effect only for the resonant case.

There are three equilibrium points for Eqs. (1-3), i.e., 1) $\theta = \pi/2$, ϕ = arbitrary value, $s = 0$; 2) $\theta = 0$, ϕ = arbitrary value, $s = -(\omega_g/\omega_n)^2$; and 3) $\theta = \pi$, ϕ = arbitrary value, $s = (\omega_g/\omega_n)^2$. We consider the motion of the rotor in the neighborhood of the first equilibrium point. The approximate solutions of Eqs. (1-3) are assumed in the form of Eqs. (40) and the damping factor is in the form of Eq. (41). Using the method of multiple scales and following the same procedure as mentioned in the preceding sections, the solutions of θ_1, ϕ_1, and s_0 for the case of which $\eta < (1+\alpha^2)^{1/2}$ are[16]

$$\theta_1 = \left\{\frac{-(1+i\alpha)b_2(T_2)}{\Lambda}\exp\left[\Lambda\frac{C}{A}T_1/2\right] - \frac{(1+i\alpha)b_2'(T_2)}{\bar\Lambda}\exp\left[\bar\Lambda\frac{C}{A}T_1/2\right]\right\}e^{i(C/A)T_0} + b_3 + \text{cc} \tag{58a}$$

$$\phi_1 = \left\{\frac{i(1+i\alpha)b_2(T_2)}{\Lambda}\exp\left[\Lambda\frac{C}{A}T_1/2\right] + \frac{i(1+i\alpha)b_2'(T_2)}{\bar\Lambda}\exp\left[\bar\Lambda\frac{C}{A}T_1/2\right]\right\}e^{i(C/A)T_0} + b_4 + \text{cc} \tag{58b}$$

$$s_0 = \left\{b_2(T_2)\exp\left[\Lambda\frac{C}{A}T_1/2\right] + b_2'(T_2)\exp\left[\bar\Lambda\frac{C}{A}T_1/2\right]\right\} \times e^{i(C/A)T_0} + \text{cc} \tag{58c}$$

where

$$\alpha = \left(\frac{\omega_g}{\Omega}\right)^2\left(\frac{A}{C}\right)^2 \tag{59a}$$

$$\Lambda = -\eta + i\sqrt{1+\alpha^2-\eta^2} \tag{59b}$$

and $\bar\Lambda$ is the complex conjugate of Λ. The solutions of θ_2, ϕ_2, and s_1 are

$$\theta_2 = -\frac{i}{2}\left\{b_2(T_2)\exp\left[\Lambda\frac{C}{A}T_1/2\right] + b_2'(T_2)\exp\left[\bar\Lambda\frac{C}{A}T_1/2\right]\right\}e^{i(C/A)T_0} + \text{cc} \tag{60a}$$

$$\phi_2 = -\alpha\left\{b_2(T_2)\exp\left[\Lambda\frac{C}{A}T_1/2\right] + b_2'(T_2)\exp\left[\bar\Lambda\frac{C}{A}T_1/2\right]\right\}e^{i(C/A)T_0} + \text{cc} \tag{60b}$$

$$s_1 = \alpha b_3(T_2) + \text{cc} \tag{60c}$$

By eliminating the secular terms in the equations governing θ_3, ϕ_3, and s_3, b_3 and b_4 in Eqs. (58) are obtained in the following form[16]:

$$b_3(T_2) = \bar b_3 \tag{61}$$

$$b_4(T_2) = \frac{A}{C}\left(\frac{\omega_g}{\Omega}\right)^2\alpha\bar b_3 T_2 + \bar b_4 \tag{62}$$

where $\bar b_3$ and $\bar b_4$ are integration constants. The solutions of b_2 and b_2' appearing in Eqs. (58) and (60) are harmonic with the exponentially decaying envelope in T_2 scale.

Decay Time Constant

The decay time constant τ_c up to the order of $T_1(=\epsilon\tau)$ is determined from the solutions of θ_1 and ϕ_1. For the underdamped case, i.e., $\eta < [1+\alpha^2]^{1/2}$,

$$\tau_c = \frac{2}{\epsilon^{1/2}\eta}\frac{A}{C} \tag{63}$$

For the critical damping case, i.e., $\eta = [1+\alpha^2]^{1/2}$,

$$\tau_c = \frac{2}{\epsilon^{1/2}(1+\alpha^2)^{1/2}}\frac{A}{C} \tag{64}$$

For the overdamped case, i.e., $\eta > [1+\alpha^2]^{1/2}$,

$$\tau_c = \frac{2}{\epsilon^{1/2}\left[\eta - \sqrt{\eta^2-1-\alpha^2}\right]}\frac{A}{C} \tag{65}$$

Residual Precession

When $\tau \to \infty$, the exponential terms in Eqs. (58) and (60) approach zero and the approximate solutions θ, ϕ, and s in the form of Eqs. (40) will become

$$\theta = \pi/2 + 2\epsilon^{1/2}\text{Re}(\bar b_3) \tag{66a}$$

$$\phi = 2\epsilon^{1/2}\frac{A}{C}\left(\frac{\omega_g}{\Omega}\right)^2\alpha\text{Re}(\bar b_3)T_2 + 2\epsilon^{1/2}\text{Re}(\bar b_4) \tag{66b}$$

$$s = 2\epsilon^{1/2}\alpha\text{Re}(\bar b_3) \tag{66c}$$

where Re(b) represents the real part of the complex number b. If the initial conditions are such that Re($\bar b_3$) is not equal to zero, the steady-state solutions will be that θ = const, s = const, and ϕ grows unboundedly as time proceeds. Substituting the steady-state solutions $\dot s = 0$ and $\dot\phi$ = const into Eq. (3), we have

$$0 = -\epsilon\left(\frac{\omega_n}{\Omega}\right)^2 s - \epsilon\left(\frac{\omega_g}{\Omega}\right)^2\cos\theta \tag{67}$$

Substituting Eq. (66a) into Eq. (67) and assuming that $2\epsilon^{1/2}\mathrm{Re}(\bar{b}_3)$ is small yields

$$s \approx 2\epsilon^{1/2}\left(\frac{\omega_g}{\Omega}\right)^2\left(\frac{\Omega}{\omega_n}\right)^2\mathrm{Re}(\bar{b}_3) = 2\epsilon^{1/2}\left(\frac{\omega_g}{\Omega}\right)^2\left(\frac{A}{C}\right)^2\mathrm{Re}(\bar{b}_3)$$

$$= 2\epsilon^{1/2}\alpha\,\mathrm{Re}(\bar{b}_3) \tag{68}$$

where Ω/ω_n is replaced by A/C in the case the the gyroscope is tuned. Equations (68) are exactly the same as Eq. (66c). In other words, since the steady-state solution of θ is not zero and due to the gravitational effect, the ball inside the damper is out of its natural position by the amount s, which is predicted theoretically by Eq. (66c). It is well known that the steady precession of a symmetric top spinning with its apex touching a point fixed in space is[14]

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{mgl}{C\Omega} \tag{69}$$

where m, C, and Ω are the mass, spin moment of inertia, and spin rate of the top, respectively; l is the distance of the center of mass of the top to its apex. The motion behavior of the two-degree-of-freedom gyroscope is similar to that of the top. Therefore, the steady precession of the two-degree-of-freedom gyroscope must be the same as Eq. (69) except that the l in Eq. (69) is replaced by $\bar{s}$, which is the steady-state displacement of the damper mass. In nondimensional form, Eq. (69) is

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{ml^2}{A}\frac{A}{C}\frac{g/l}{\Omega^2}\frac{\bar{s}}{l} = \epsilon\left(\frac{A}{C}\right)\left(\frac{\omega_g}{\Omega}\right)^2 s \tag{70}$$

The substitution of Eq. (66b) into Eq. (70) for s yields

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 2\epsilon^{3/2}\left(\frac{A}{C}\right)\left(\frac{\omega_g}{\Omega}\right)^2\alpha\,\mathrm{Re}(\bar{b}_3) \tag{71}$$

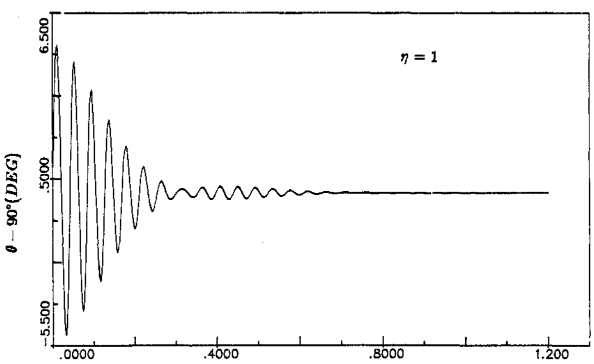

Fig. 6 Time history of the nutation angle for $\eta = 1$ with gravity effect.

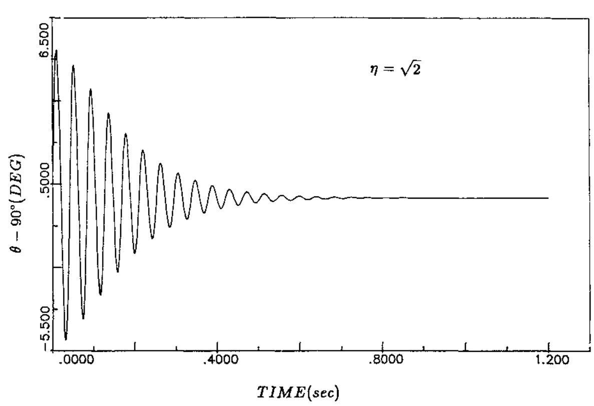

Fig. 7 Time history of the nutation angle for $\eta = \sqrt{2}$ with gravity effect.

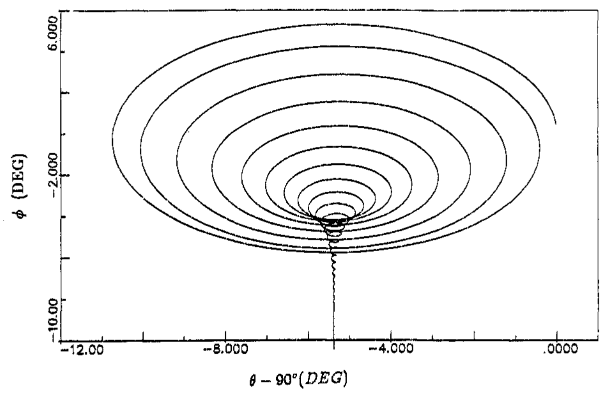

Fig. 8 Trajectory of the rotor in $\phi, \theta^*(=\theta - 90$ deg) plane.

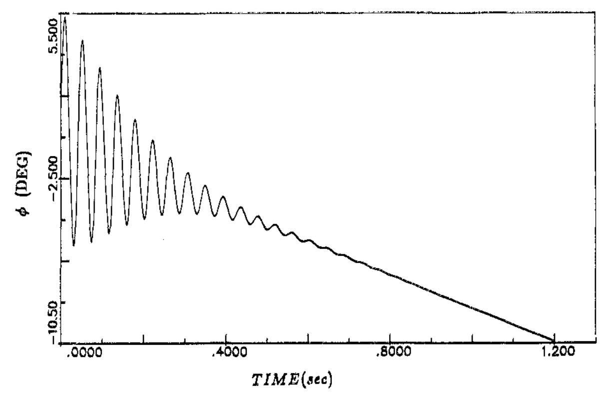

Fig. 9 Residual precession due to the effect of gravity.

Equation (71) is equal to the differentiation of Eq. (66b) with respect to τ, which verifies the validity of our approximate solutions again.

Numerical Results and Discussions

Consider the special example for which the parameters of the tuned gyroscope are $\epsilon = 0.01$, $C/A = 1.5$, $\omega_g = 1.5 \times 10^2$ rad/s, $\Omega = 1.0 \times 10^2$ rad/s, and $\omega_n = 1.5 \times 10^2$ rad/s. Then, from Eq. (63), $\alpha = 1$. The initial conditions are all zero except that $\theta(0) = 90$ deg and $\dot{\theta}(0) = 800$ deg/s. By integrating the exact equations of motion, the time history of the nutation angle for various values of η are shown in Figs. 6 and 7. With gravity, the critical damping does not occur at $\eta = 1$ but at $\eta = \sqrt{2}$, as shown in Fig. 7, which is consistent with the predicted value, i.e., $\eta = \sqrt{1 + \alpha^2}$, by the perturbation solutions. For the same system parameters, if the initial conditions are changed to that $\theta(0) = 90$ deg, $\dot{\phi}(0) = 800$ deg/s, and the others are zero, from numerical solutions the trajectory of the rotor for the critical damping case in the $\phi, \theta^*(=\theta - 90$ deg) plane is shown in Fig. 8. As it can be seen, in steady state, θ is constant and ϕ seems to grow unbounded. The numerical solution of $\phi(t)$ is shown in Fig. 9. It is clear that in steady state ϕ is a linear function of time and the value of the precession $\dot{\phi}$ is consistent with the perturbation solution given by Eq. (71).

Gravity has the effect of enhancing the damping performance of the damper, so the energy of the coning motion of the rotor is dissipated faster. This can be seen by comparing Eqs. (63–65) with Eqs. (53–55) because the decay time constant with gravity considered is less than that with gravity removed. Since for certain sets of initial conditions the gyroscope has inevitably residual precession, the gyroscope with the damper oriented parallel to the spin axis of the rotor as shown in Fig. 1 is undesirable. If the damper is placed in the segment of the inner gimbal which is perpendicular to the spin axle and a balance mass is placed in the opposite segment, the gravity regardless of its direciton will have no effect on the motion of the gyroscope. In this case, the residual precession does not exist, the equations of motion are the same as Eqs. (1-3) with gravity removed, and the decay time constant is identical to one of Eqs. (53-55) for the tuned gyroscope.

Conclusions

The damping performance of a ball-in-tube damper, which is used to remove the coning motion of a two-degree-of-freedom gyroscope, was analyzed in this paper. An analytical expression for the time constant was obtained by using the method of multiple scales. The results from the analytical solutions show that the removal of the coning motion can be improved in the following ways: first, the natural frequency of the ball-in-tube damper should be tuned to equal the nutation frequency of the gyroscope without the damper; second, the damping coefficient should be chosen close to the minimum point of the curve of the decay time constant vs the damping factor. The damper placed inside the inner gimbal must be so oriented that the damper axis is perpendicular to the spin axis of the rotor; otherwise the rotor will have unwanted residual precession.

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